Compactified Imaginary Liouville theory

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Quantization days 4.0

Based on joint works with C. Guillarmou & A. Kupiainen

Path integral

Riemann surface Σ, Riemannian metric g

$$\langle F \rangle_{\Sigma,g} := \int_{\Phi:\Sigma\to\mathbb{R}/2\pi R\mathbb{Z}} F(\Phi) e^{-S_\Sigma(\Phi,g)} D\Phi$$

Liouville action

$$S_\Sigma(\Phi,g) = \frac{1}{4\pi}\int_\Sigma (|d\Phi|_g^2 + iQK_g\Phi + \mu e^{i\beta\Phi})\mathrm{dv}_g$$

Parameters

$$\beta \in (0,2), \quad Q = \frac{\beta}{2} - \frac{2}{\beta}, \quad \mu \in \mathbb{C}$$

CFT with central charge $\mathbf{c} = 1 - 6Q^2$

Why CILT?

Conjectural scaling limit of **Loop models**

- e.g. critical Potts or $O(n)$ (Jacobsen, Kondev, Saleur, etc...)
- applications to conformal critical curves (Binder, Duplantier, Gruzberg, Wiegmann, etc...)

Maths challenges :

- Probabilistic methods for non-unitary CFTs (Logarithmic CFT)
- Higher order renormalisation schemes (similar to Sine-Gordon)
- Minimal models: path integral reformulation of BRST cohomology (Felder)

Main results

CILT path integral

$$\langle F\rangle_{\Sigma,g} := \int_{\Phi:\Sigma\to\mathbb{R}/2\pi R\mathbb{Z}} F(\Phi) e^{-\frac{1}{4\pi}\int_\Sigma (|d\Phi|_g^2 + iQK_g\Phi + \mu e^{i\beta\Phi})\mathrm{dv}_g} D\Phi.$$

Theorem (GKR '23)

Fix $\beta^2 < 2$ and set $Q = \frac{\beta}{2} - \frac{2}{\beta}$.

Rational case: $\beta^2 \in \mathbb{Q}$.

One can construct the path integral and correlation functions on any Riemann surface Σ. It obeys Segal's axioms of CFT with central charge $1 - 6Q^2$.

Irrational case: $\beta^2 \notin \mathbb{Q}$.

Same conclusion on surfaces of genus 0 or 1.

Hints for the construction

Segal's axioms and bootstrap

Compactified GFF

Riemann surface Σ, Riemannian metric g

$$F \mapsto \int_{\Phi:\Sigma\to\mathbb{R}/2\pi R\mathbb{Z}} F(\Phi) e^{-\frac{1}{4\pi}\int_\Sigma |d\Phi|_g^2 \mathrm{dv}_g} D\Phi$$

If $\Phi : \Sigma \to \mathbb{R}/2\pi R\mathbb{Z}$ then $d\Phi$ is a closed 1-form on Σ.
Hodge decomposition on Σ

Closed 1-forms = Exact 1-forms $\oplus$ De Rham cohomology

The integration measures splits in two parts

- on exact forms, (standard) Gaussian Free Field
- discrete Gaussian on the De Rham cohomology group.
  Remark: this group is trivial on the Riemann sphere (this talk)

Gaussian Free Field

Let X_g be the GFF on Σ in the metric g on Σ

$$X_g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \geqslant 1} \frac{\alpha_n}{\sqrt{\lambda_n}} e_n(x)$$

with

- $(\alpha_n)_n$ iid standard Gaussians
- $(e_n)_n$ orthonormal basis of eigenfunctions of Laplacian Δ_g with eigenvalues $(\lambda_n)_n$ and b.c. $\int_\Sigma e_n \mathrm{dv}_g = 0$
- Covariance $\mathbb{E}[X_g(x)X_g(x')] = G_g(x, x')$ Green function of the Laplacian.

The Riemann sphere

Let Σ be the Riemann sphere with a metric g.

Gaussian integral

$$\int_{\Phi:\Sigma\to\mathbb{R}/2\pi R\mathbb{Z}} F(\Phi) e^{-\frac{1}{4\pi}\int_\Sigma |d\Phi|_g^2 \mathrm{dv}_g} D\Phi := C(g) \int_{\mathbb{R}/2\pi R\mathbb{Z}} \mathbb{E}\Big[F(c+X_g)\Big]\mathrm{d}c$$

with $C(g) := \left(\frac{\mathrm{v}_g(\Sigma)}{\det'(\Delta_g)}\right)^{\frac{1}{2}}$.

The Riemann sphere

Imaginary Liouville path integral

$$\langle F \rangle_{\Sigma,g} := C(g) \int_{\mathbb{R}/2\pi R\mathbb{Z}} \mathbb{E}\Big[F(c+X_g) e^{-\frac{iQ}{4\pi}\int_\Sigma K_g(c+X_g)\,\mathrm{d}v_g - \mu \int_\Sigma e^{i\beta(c+X_g)}\mathrm{d}v_g} \Big] \mathrm{d}c$$

Imaginary Gaussian multiplicative chaos (GMC, Lacoin-Rhodes-Vargas '19):

$$\int_\Sigma e^{i\beta X_g}\mathrm{d}v_g := \lim_{\epsilon\to 0} \epsilon^{-\frac{\beta^2}{2}} \int_\Sigma e^{i\beta X_{g,\epsilon}}\mathrm{d}v_g$$

is non trivial for $\beta^2 < 2$ and has exponential moments of all orders.

Remarks:

1) for the potential and curvature to be functions on the circle, we must have $\beta^2 \in \mathbb{Q}$

2) Sine-Gordon like renormalisation for $2 \leqslant \beta^2 < 4$

Correlation functions of electric operators

Electric operators:

$$V_\alpha(x) := e^{i\alpha\Phi(x)}, \qquad \text{for } x \in \Sigma,\ \alpha \in \frac{1}{R}\mathbb{Z}$$

Correlation functions: $\mathbf{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$

$$\langle V_{\boldsymbol{\alpha}}(\mathbf{x}) \rangle_{\Sigma,g} := \langle V_{\alpha_1}(x_1) \ldots V_{\alpha_n}(x_n) \rangle_{\Sigma,g}$$

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Theorem

If $\alpha_j > Q$ and $\alpha_j \in \frac{1}{R}\mathbb{Z}$ for all j, then

$$\langle V_{\boldsymbol{\alpha}}(\mathbf{x})\rangle_{\Sigma,g} = C(g,\mathbf{x}) \int_{\mathbb{R}/2\pi R\mathbb{Z}} \mathbb{E}\Big[e^{-\frac{iQ}{4\pi}\int_\Sigma K_g(c+X_g+u_{\mathbf{x}})\,\mathrm{d}v_g - \mu \int_\Sigma e^{i\beta(c+X_g+u_{\mathbf{x}})}\mathrm{d}v_g}\Big]\mathrm{d}c,$$

with $u_{\mathbf{x}}(z) := \sum_{j=1}^n i\alpha_j G_g(z, x_j)$, is well defined.

Correlation functions of magnetic operators

Magnetic operators:

$$O_m(z), \qquad \text{for } m \in \mathbb{Z}, z \in \Sigma$$

has the effect of forcing the field Φ to have a winding $2\pi Rm$ around z.

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Construction: $\mathbf{z} = (z_1, \ldots, z_p)$, assume $\sum_{j=1}^{p} m_j = 0$.
Consider a harmonic 1-form $\omega_{\mathbf{z},\mathbf{m}}$ on Σ s.t.

$$\int_{\gamma(z_j)} \omega_{\mathbf{z},\mathbf{m}} = 2\pi R m_j$$

with $\gamma(z_j)$ a small contour surrounding z_j.

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$$\Phi_g := c + X_g + I_{x_0}(\omega_{\mathbf{z},\mathbf{m}}) \qquad \text{and} \qquad I_{x_0}(\omega_{\mathbf{z},\mathbf{m}})(x) := \int_{x_0}^{x} \omega_{\mathbf{z},\mathbf{m}}$$

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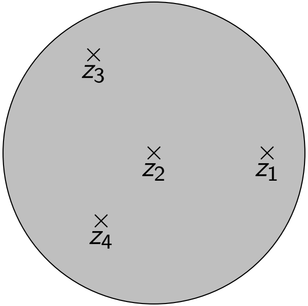
Multivalued: we need to construct a branch cut

Defect graph

Consider unit tangent vectors $v_j \in T_{x_j}\Sigma$ and $p-1$ smooth arcs $\zeta_\ell : [0,1] \to \Sigma$ such that

- the arcs connect all the points and don't intersect
- the arcs leave and arrive points along the unit vectors
- arcs are oriented in the sense of increasing charges
- no cycle, connected



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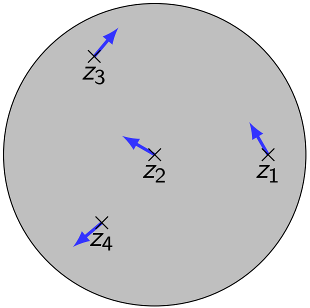

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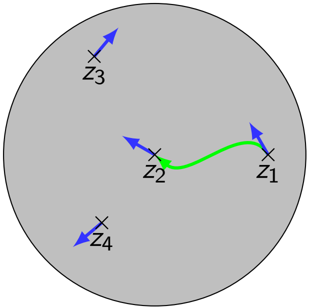

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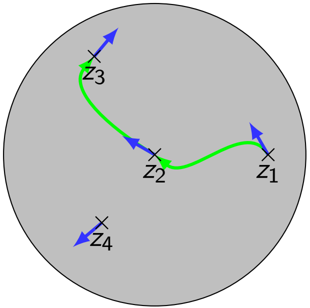

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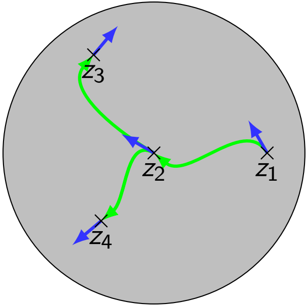

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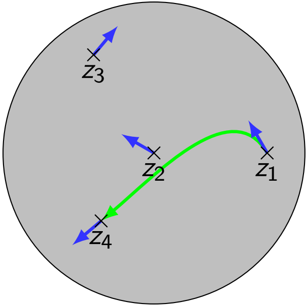

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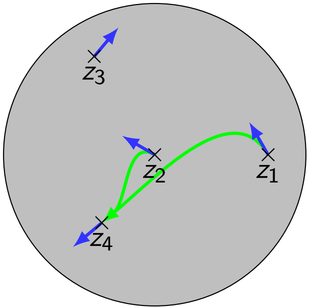

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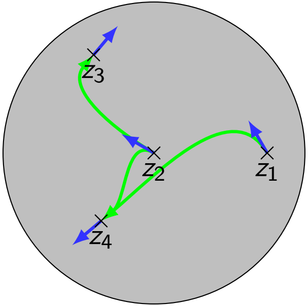

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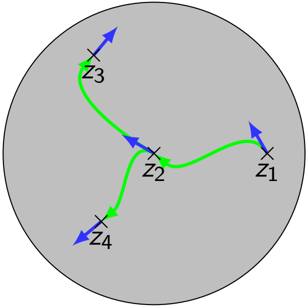
Defect graph and regularized curvature

Regularized curvature:

$$\int_{\Sigma}^{\mathrm{reg}} K_g I_{x_0}(\omega_{\mathbf{z},\mathbf{m}})\,\mathrm{d}v_g := \int_{\Sigma\setminus\mathcal{D}_{\mathbf{z},\mathbf{v}}} K_g I_{x_0}(\omega_{\mathbf{z},\mathbf{m}})\,\mathrm{d}v_g - 2\sum_{\ell=1}^{p-1}\kappa(\xi_\ell)\int_{\xi_\ell} k_g \mathrm{d}\ell_g$$



$\kappa(\xi_\ell) := \sum$ charges in the yellow area $= m_1 + m_2 + m_3$

Theorem:

The regularized curvature does not depend on the defect graph, only on the points z_j, the charges m_j and vectors v_j.

Proof: Gauss-Bonnet

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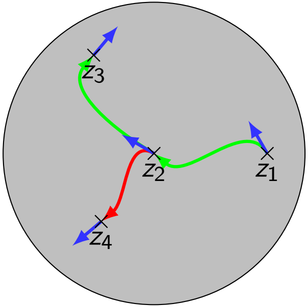

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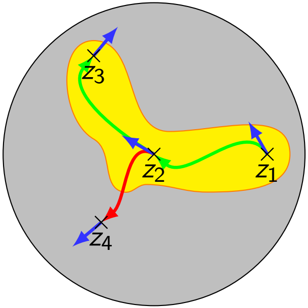

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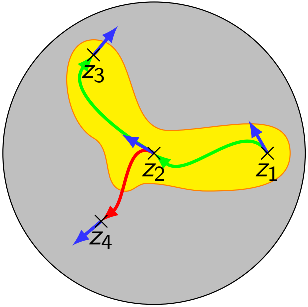

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Magnetic operators and regularized curvature

$$\langle O_{\mathbf{m}}(\mathbf{z},\mathbf{v})\rangle_{\Sigma,g} := C(g)\int_{\mathbb{R}/2\pi R\mathbb{Z}} \mathbb{E}\Big[e^{-\frac{iQ}{4\pi}\int_{\Sigma}^{\mathrm{reg}} K_g \Phi_g \,\mathrm{d}v_g - \mu\int_{\Sigma} e^{i\beta\Phi_g}\mathrm{d}v_g}\Big]\mathrm{d}c$$

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Theorem:

The path integral does not depend on the defect graph, only on the points z_j, the charges m_j and the vectors v_j.

Electro-magnetic operators

$$\langle V_{\boldsymbol{\alpha},\mathbf{m}}(\mathbf{z},\mathbf{v})\rangle_{\Sigma,g} := \lim_{\substack{x_j \to z_j \\ v_j}} \langle V_{\boldsymbol{\alpha}}(\mathbf{x}) O_{\mathbf{m}}(\mathbf{z},\mathbf{v})\rangle_{\Sigma,g}$$

Theorem: Well defined if $\sum_j m_j = 0$ and $\forall j,\ \alpha_j \in \frac{1}{R}\mathbb{Z}$. They obey

- **diffeomorphism invariance**: if $\psi : \Sigma' \to \Sigma$ diffeo

$$\langle V_{\boldsymbol{\alpha},\mathbf{m}}(\psi(\mathbf{z}),\psi_*\mathbf{v})\rangle_{\Sigma,g} = \langle V_{\boldsymbol{\alpha},\mathbf{m}}(\mathbf{z},\mathbf{v})\rangle_{\Sigma',\psi^*g}$$

- **local scale covariance:** if $g' = e^{\varphi}g$ for $\varphi : \Sigma \to \mathbb{R}$ smooth then

$$\langle V_{\boldsymbol{\alpha},\mathbf{m}}(\mathbf{z},\mathbf{v})\rangle_{\Sigma,e^{\varphi}g} = e^{\frac{\mathbf{c}_L}{96\pi}\int_\Sigma |d\varphi|_g^2 + 2K_g\varphi}\Big(e^{-\sum_j \Delta_{\alpha_j,m_j}\varphi(z_j)}\Big)\langle V_{\boldsymbol{\alpha},\mathbf{m}}(\mathbf{z},\mathbf{v})\rangle_{\Sigma,g}$$

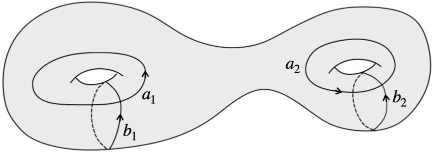
with central charge $\mathbf{c}_L = 1 - 6Q^2$ and conf. weight $\Delta_{(\alpha,m)} = \frac{\alpha}{2}(\frac{\alpha}{2} - Q) + \frac{m^2R^2}{4}$.

- **Spin:** if r_θ stands for the rotation of angle θ

$$\langle V_{\boldsymbol{\alpha},\mathbf{m}}(\mathbf{z},r_{\boldsymbol{\theta}}\mathbf{v})\rangle_{\Sigma,g} = e^{iR(\boldsymbol{\alpha}-Q)\langle \mathbf{m},\boldsymbol{\theta}\rangle}\langle V_{\boldsymbol{\alpha},\mathbf{m}}(\mathbf{z},\mathbf{v})\rangle_{\Sigma,g}$$

On general Riemann surfaces

Compactified GFF has a random topological component: instantons



Further regularisation of curvature term needed due to instantons

This regularisation has a peculiar behaviour under the action of the symplectic group which induces a distinction between $\beta^2 \in \mathbb{Q}$ and $\beta^2 \notin \mathbb{Q}$

Hints for the construction

Segal's axioms and bootstrap

Comments on the conformal bootstrap

Conformal bootstrap aims to compute

$$\langle V_{\boldsymbol{\alpha},\mathbf{m}}(\mathbf{z},\mathbf{v})\rangle_{\Sigma,g}$$

Implementing the conformal bootstrap requires at least to determine two datas:

- the **structure constants**, namely 3 point correlation on the Riemann sphere
- the **spectrum**: "eigenstates" of the Hamiltonian

Computing the correlation of the CFT is then a (nontrivial) Plancherel type formula based on the spectrum with structure constants as coefficients

Structure constants (3 point correlation functions)

On the extended complex plane (with e_1 the unit vector parallel to x axis)

$$\langle V_{\alpha_1,m_1}(0,e_1)V_{\alpha_2,m_2}(1,e_1)V_{\alpha_3,m_3}(\infty,e_1)\rangle.$$

If $\alpha_1+\alpha_2+\alpha_3=2Q-n\beta$ ($n\in\mathbb{N}$) and $\Delta_i,\bar{\Delta}_i=\beta\alpha_i\pm\frac{R\beta m_i}{2}$ ($i=1,2$)

$$2\pi R\frac{(-\mu)^n}{n!}\int_{\mathbb{C}^n}\prod_{j=1}^{n}x_j^{\Delta_1}\bar{x}_j^{\bar{\Delta}_1}(1-x_j)^{\Delta_2}(1-\bar{x}_j)^{\bar{\Delta}_2}\prod_{j<j'}|x_j-x_{j'}|^{\beta^2}\mathrm{d}x_1\ldots\mathrm{d}x_n$$

It s a square root of **imaginary DOZZ** (Zamolod. '05)

$$\langle V_{\alpha_1,m_1}(0,e_1)V_{\alpha_2,m_2}(1,e_1)V_{\alpha_3,m_3}(\infty,e_1)\rangle^2=4\pi^2R^2\mu^{\frac{2}{\beta}(2Q-\sum\alpha_i)}$$
$$C_\beta^{\mathrm{DOZZ}}(\alpha_1+m_1R,\alpha_2+m_2R,\alpha_3+m_3R)C_\beta^{\mathrm{DOZZ}}(\alpha_1-m_1R,\alpha_2-m_2R,\alpha_3-m_3R).$$

Structure constants (3 point correlation functions)

On the extended complex plane (with e_1 the unit vector parallel to x axis)

$$\langle V_{\alpha_1,m_1}(0,e_1)V_{\alpha_2,m_2}(1,e_1)V_{\alpha_3,m_3}(\infty,e_1)\rangle.$$

If $\alpha_1+\alpha_2+\alpha_3=2Q-n\beta$ ($n\in\mathbb{N}$) and $\Delta_i,\bar{\Delta}_i=\beta\alpha_i\pm\frac{R\beta m_i}{2}$ ($i=1,2$)

$$2\pi R\frac{(-\mu)^n}{n!}\int_{\mathbb{C}^n}\prod_{j=1}^{n}x_j^{\Delta_1}\bar{x}_j^{\bar{\Delta}_1}(1-x_j)^{\Delta_2}(1-\bar{x}_j)^{\bar{\Delta}_2}\prod_{j<j'}|x_j-x_{j'}|^{\beta^2}\mathrm{d}x_1\ldots\mathrm{d}x_n$$

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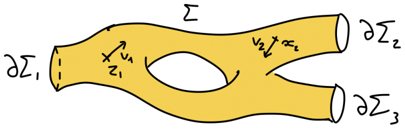
$$C_\beta^{\mathrm{DOZZ}}(\alpha_1+m_1R,\alpha_2+m_2R,\alpha_3+m_3R)C_\beta^{\mathrm{DOZZ}}(\alpha_1-m_1R,\alpha_2-m_2R,\alpha_3-m_3R).$$

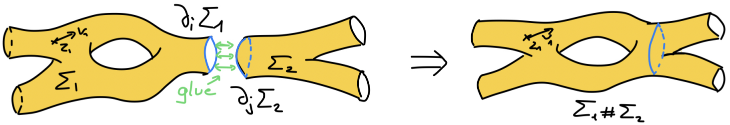
Segal's axioms

The path integral definition can be extended to **amplitudes**, namely surfaces with boundary

$$\mathcal{A}_{\Sigma,g,\mathbf{z},\mathbf{v},\boldsymbol{\alpha}}(\varphi) := \int_{\substack{\{\Phi:\Sigma\to\mathbb{R}/2\pi R\mathbb{Z},\\ \Phi|_{\partial\Sigma_i}=\varphi_i\}}} V_{\boldsymbol{\alpha},\mathbf{m}}(\mathbf{z},\mathbf{v})\, e^{-S_\Sigma(\Phi,g)}\, D\Phi$$



Checking Segal's axioms consists in proving that the amplitudes behave well under gluing along their boundaries



Gluing amplitudes involves integration of the boundary fields in some Hilbert space

Hilbert space

Hilbert space:

$$\mathcal{H} := L^2\big(\mathbb{Z} \times (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}^2)^{\mathbb{N}_0}, \mu\big)$$

with

$$\mu := \mu_{\mathbb{Z}} \otimes \mathrm{d}c \otimes \mathbb{P}$$

with $\mu_{\mathbb{Z}}$ counting measure on $\mathbb{Z}$, $\mathrm{d}c$ Leb. measure on circle, $\mathbb{P}$ law of i.i.d. standard Gaussians on $(\mathbb{R}^2)^{\mathbb{N}_0}$.

Coordinate map

$$\big(k, c, (x_n, y_n)_n\big) \quad \mapsto \quad \varphi(\theta) := c + kR\theta + \sum_{n>0} \frac{x_n}{\sqrt{n}} \cos(n\theta) - \frac{y_n}{\sqrt{n}} \sin(n\theta)$$

Hamiltonian

Considering gluing of annuli amplitudes, we obtain a semigroup, with generator called the **Hamiltonian** of the CFT.

Hilbert space: $\mathcal{H} := L^2\big(\mathbb{Z} \times (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}^2)^{\mathbb{N}_0}, \mu\big)$

Coordinate map

$$\varphi(\theta) := c + kR\theta + \sum_{n>0} \frac{x_n}{\sqrt{n}} \cos(n\theta) - \frac{y_n}{\sqrt{n}} \sin(n\theta)$$

Hamiltonian:

$$\mathbf{H} := -\tfrac{1}{2}\partial_c^2 - \tfrac{Q^2}{2} + \mathbf{P} + \tfrac{1}{2}R^2k^2 + \mu \int_0^{2\pi} e^{i\beta\varphi(\theta)}\,\mathrm{d}\theta$$

with

$$\mathbf{P} := \sum_{n>0} n(x_n\partial_{x_n} - \partial_{x_n}^2 + y_n\partial_{y_n} - \partial_{y_n}^2)$$

Finding the spectrum of the CFT consists in "diagonalizing" the Hamiltonian

Questions

Hamiltonian and spectrum

- Non self-adoint
- "Eigenstates" form Jordan cells $\Rightarrow$ Log CFT
- Role of singular vectors?

Link with minimal models

- BRST cohomology (Felder)
- link between unitary minimal models and SG thresholds?

Link with Loop models

- Generating critical curves
- Connection with SLE/CLE?